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The Cause Of The Precession Of The Perihelion Of Mercury And Of Elliptical Orbits

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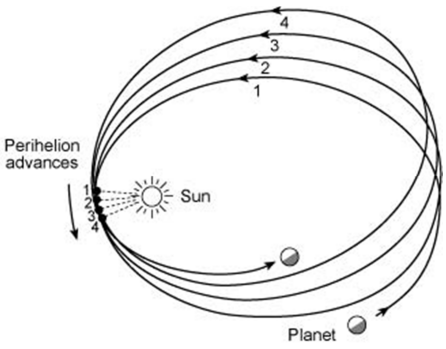
Introduction

In 1846 the French mathematician, Urbain Le Verrier, using Newton's laws predicted the existence of Neptune to a high degree of precision, after analysing perturbations observed in the movement of the planet Uranus. Following this monumental success he turned his attention to the motion of Mercury.



Mercury follows an elliptical path and is the closest planet to the sun. It was already known that the axis of Mercury's ellipse precesses as shown. To be absolutely clear about the precession, the orbital path is not rising out of the page, like a lob sided helix, it is rolling to the left on the plane of the page. In 1859, after many years of careful research, Le Verrier presented his paper. It concluded that the perihelion was precessing faster than could be explained using Newton's laws, by 38 seconds of one minute of one degree every 100 years. It is now generally considered to be approximately 43.1" seconds. Mercury precesses by a total angle of approximately 574.7" per century and Newton's laws predicted 531.6" as this is caused by the effect of the outer planets, the oblateness of the Sun and by asteroids. We will discuss the precession caused by the outer planets later on. If we remove the outer planets temporarily from the solar system, according to Newton, the orbit should continuously cover the same path, over and over but as can be seen, it is continuously moving forward. Again to be completely clear, a precession means the axis rotates in the same direction as the orbit, if it went the other way, it would be a regression.

So If the solar system consisted of only the Sun and Mercury, the precession would still occur and it would be approximately 43.1" of arc per century. Newton's theory of gravity in this scenario would however predict no precession at all.

Following his discovery and to try to resolve the anomaly, Le Verrier hypothesised the existence of a planet which would lie between Mercury and the Sun. This was a similar fix to the prediction of Neptune to resolve the anomalous motion of Uranus. The proposed planet was named "Vulcan" , one might say a little prematurely as despite thorough investigation, Vulcan was never found. Vulcan was hypothesised as a single planet but also as an orbital zone of many much smaller planets.

Einstein introduced his new theory of gravity in 1915, The General theory of Relativity (GR) and this was able to provide a correct mathematical prediction for the precession. Predicting the precession of Mercury came to be known as one of the 4 classic tests of General Relativity.

The other 3 classic tests were the bending of light by the sun, gravitational redshift and the Shapiro time delay. The Shapiro time delay is the delay of a radar beam passing the sun caused by the beam being curved by gravity. The curved path being longer than a straight path causes the delay. Whilst this is a different test to the curvature of light, it is effectively testing the same thing. Gravitational Redshift was also described as a classical test of GR but is generally no longer included as a test, as no part of GR is needed to predict it. So the 4 classical tests reduce down to two phenomena: The precession of elliptical orbits and the curvature of light by gravity. Newton gravity also predicted the curvature of light, but his equations extended by Soldner and Cavendish only predicted half of the observed value.

Had Newton gravity predicted the precession of Mercury, and the curvature of light had been found to be twice that predicted by Newton's gravitation equation, two options would have been available. Either Newton gravity must have been wrong or the gravitational mass of photons must have been twice that of the inertial mass. That is so say that a photon by its nature, is affected by the force of gravity by twice as much as normal matter.
In the 1931 paper "On the gravitational field produced by light" by Richard C Tolman et al, it was concluded that Newton gravity predicts the correct bending of light if we assume photons to have double the gravitational mass compared to the inertial mass.

It is difficult to overstate the historic importance of Mercury's precession anomaly, as without it, Newton gravity with minor mathematical and philosophical changes might still have been the only credible theory of gravity today.

Newton's inverse Square law is given below where F is the force, G is Newton's Universal gravitational constant, m_1 & m_2 are the 2 masses being considered ie the Sun and the Earth and r is the distance between them.

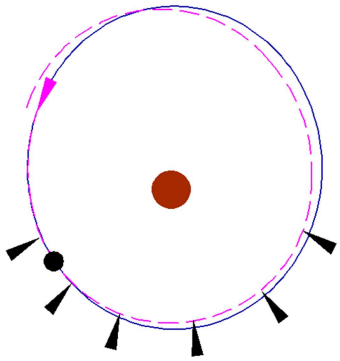
$$F = \frac{Gm_1m_2}{r^2}$$

Why Do Elliptical Orbits Precess

I am going to start by proposing an intuitive model which would cause the precession of elliptical orbits. I am then going to talk through the remedial theories proposed by Asaph Hall/Simon Newcomb, Paul Gerber and by Leverrier. Although undoubtedly those models were wrong, by analysing how they worked, it helps to strengthen our insight into the true cause of elliptical precessions. We will then move onto the theory of General Relativity and discuss whether this is the true cause of the precession.

Intuitive Model In Which Newton's Gravitational Potential Increases With Rotational Velocity

In the image below the solid blue line shows Newton's orbit which is a closed orbit and has no precession. The magenta line (dashed line for those viewing in black and white) indicates an orbit in which the gravitational potential increases slightly with velocity compared to Newton's potential. The velocity of elliptical orbits increases towards the perihelion where the radial distance from the sun reduces. The gravitational force would increase slightly relative to Newton in the region of the perihelion due to the increase in velocity and this is indicated by the arrow heads. As the force increases slightly, the radial distance will reduce slightly relative to Newton as indicated. As the planet moves away from the perihelion and the elevated velocity in the region of the right arrow, the force and the approximate shape of the orbit returns to being more in accordance with Newton. However the angle has changed from the previous orbit and so the orbit precesses. So by introducing a modification to Newton gravity in which the gravitational potential increases slightly with velocity, the orbit will precess.



Gerber's Gravity

In 1898 Paul Gerber, a German School teacher published a paper which provided the exact same equation for the perihelion precession of elliptical orbits that Einstein was to produce 17 years later.

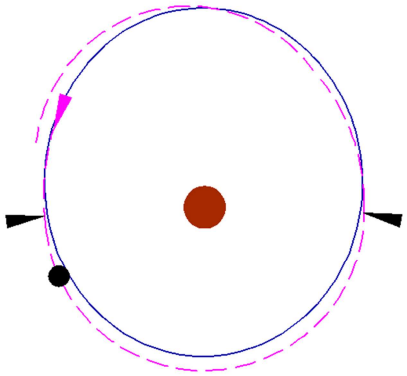
The precession equation derived by Gerber and later by Einstein is given below. G is Newton's Universal Gravitational constant. m' is the mass of the sun, c is the speed of light, r is the semi major axis of the ellipse and e is the eccentricity of the orbit.

$$\Delta\phi = \frac{6\pi Gm'}{c^2 r(1-e^2)}$$

In his theory, the potential of gravity changed slightly with the **radial** velocity of an orbit. How Gerber arrived at his potential has been questioned by many and is generally considered to be flawed. Nevertheless, if we ignore how he derived it, the potential does lead to the correct precession equation and so we will look at why this is.

Gerber's Potential is: $a = \frac{GM}{r(1 - \dot{r}/c)^2}$

$\dot{r}$ is the radial distance from the sun to the planet. As the radial distance decreases, when the planet approaches the perihelion, the gravitational force reduces compared to Newton and when the planet moves away from the perihelion and the radial distance increases, so the force increases relative to Newton. This relative reduction in force and subsequent increase in force means that the orbital distance will increase relative to Newton when approaching the perihelion and reduce when leaving the perihelion as indicated below. ie it swings out slightly and then swings in again. Whilst the radial distance is almost identical at the right arrow to the earlier distance at the left arrow, the angle is different compared to Newton and so each next orbit starts with a different angle which causes the precession to occur.



Some Scientists including Roseveare in his 1979 paper, "Leverrier To Einstein A Review Of The Mercury Problem" have interpreted Gerber's potential in the opposite way. That would cause the orbit to sway in slightly and then sway out again. Instead of causing a precession, that would cause a regression and so it could not be interpreted that way.

Asaph Hall / Simon Newcomb's Theory

In 1894 Asaph Hall, an American astronomer published a paper titled "A suggestion In The Theory Of Mercury" in which he adjusted the r^2 element of Newton's Inverse Square law to approximately $r^{200000016}$. In doing this, he substantially predicted the precession of Mercury but overly predicted the precession of the Earth, Venus and Mars.

Before discussing the modification proposed by Asaph Hall, I should point out again that Newton gravity does not predict any precession in elliptical orbits. In addition, there needs to be an inconsistent change to the force compared to Newton, to produce a precession or regression. If we were to simply change the mass of the Sun or to change Newton's Gravitational constant G to a different figure, we would not get any precession. Instead the orbital radius would either increase or decrease but there would be no precession. If for example you consistently increase the curvature of a circle throughout its circumference, you would simply get a smaller circle. The same would apply to an ellipse. So to get a precession, you have to increase the curvature for a part of the orbit and in the right location but not for all of it otherwise you simply change the size, not the shape.

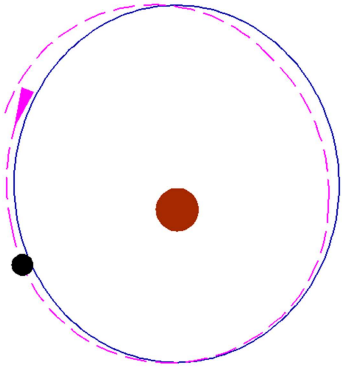
Asaph Hall proposed a modification of Newton's inverse square law and this was researched and analysed thoroughly by Simon Newcomb around 1895. Simon Newcomb was a Canadian-American mathematician and Astronomer. Newcomb was an extremely accomplished astronomer of his day and many of his observations are still found to be very accurate today.

The table below shows the total discrepancy of Newton's theory and the improved proposal of Newcomb and the solution from Einstein.

	Mercury	Venus	Earth	Mars
Discrepancy between observation and Newtonian theory	43.1 ±0.45	8.4 ±4.8	5.0 ±1.2	1.36
Calculated additional advance from general theory of relativity	42.98	8.61	3.84	1.35
Newcomb's attempted correction for perihelion advance	43.37	16.98	10.45	5.55

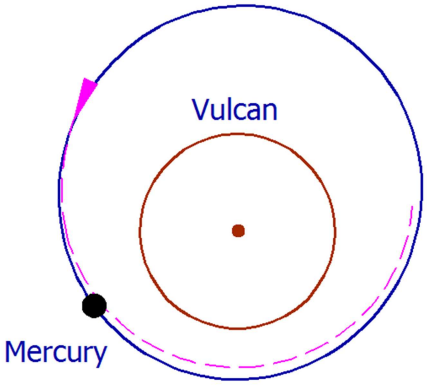
When we look at Simon Newcomb's modification of R^2, this reduces the force of gravity throughout the whole orbit but makes the force at the perihelion proportionately greater than it is for the rest of the orbit. You would therefore expect

the radial distance to be slightly greater through the entire orbit relative to Newton. However, as the force reduction would be greatest at the aphelion, there would be a proportional increase in force around the perihelion. The orbit would therefore reduce radially on the approach to and leaving from the perihelion when compared to Newton and so the direction at the start of each subsequent orbit would be angled in, slightly relative to Newton and so a precession would occur as below. The effect is therefore very similar to what I have called “The Intuitive Model”.



Vulcan

We talked earlier about the hypothetical planet Vulcan proposed by Leverrier. It’s orbit would have been located between Mercury and the Sun as indicated below. Vulcan was hypothesised as a single planet and also as a series of much smaller planets in a Vulcanoid zone. As Vulcan and Mercury would orbit at different rates, they would frequently pass each other and so whether a single planet or a zone, the effect would approximate mathematically to a zone as indicated by the red circle. You can see that Mercury will be much closer to the Vulcan orbit at the perihelion and so whilst Vulcan increases the inward pull throughout the whole orbit, it increases it most significantly at the perihelion where the velocity is greatest. It is again therefore almost identical in its affect to the “Intuitive Model”.

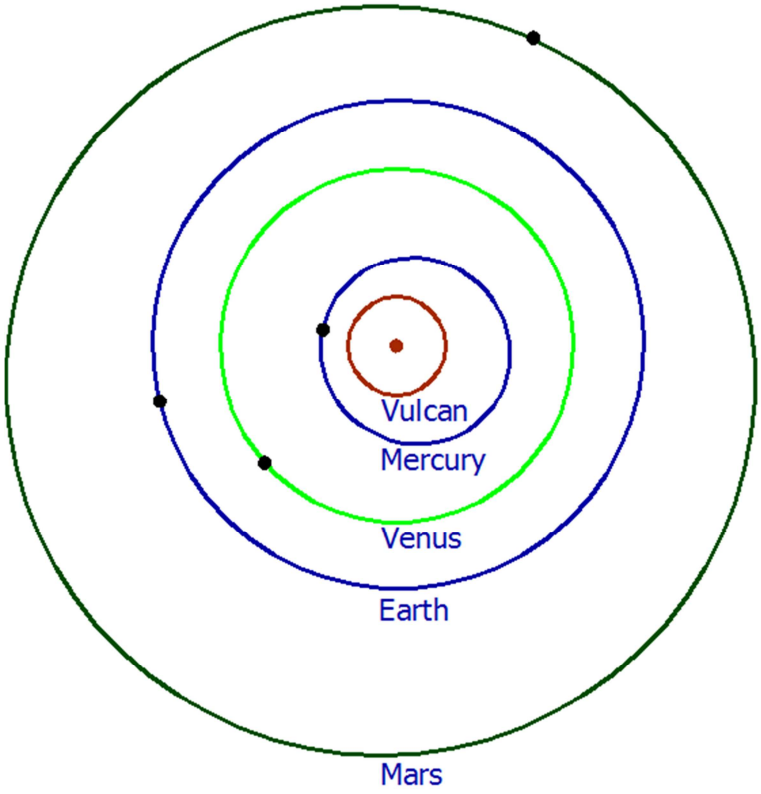


The Precession Caused By The Outer Planets

Although the table below is missing the effects of the oblateness of the Sun and the effects of the asteroids, Uranus and Neptune, these have only a very small effect and so it is a reasonable breakdown of the causes of the perihelion precession of Mercury.

Planet causing precession	Precession Angle (Seconds/Century)
Venus	277.8
Earth	90.0
Mars	2.5
Jupiter	153.6
Saturn	7.3
Total Calculated	531.2
Total Observed	574.3
Discrepancy	43.1
Post Newtonian Calculation (GR)	42.98

The image below shows the orbits of the inner planets and the planets are located on the image in the location of their respective perihelia. Whilst the perihelia are not aligned, Mercury's ellipticity is sufficiently large to ensure that the distance to each of the planet rings is greatest towards Mercury's perihelion and least towards its aphelion. It is therefore analogous to the Asaph Hall modification. The gravitational force towards the Sun is reduced by each of the outer planets throughout the whole of Mercury's orbit but to the greatest extent at the aphelion where it is closest to the rings of the outer planets. This gives a proportional increase in force at the perihelion. Even Jupiter which has a very significant precessional effect and has its perihelion offset by about 90^0 compared to Mercury, still has the effect of increasing Mercury's precession.



General Relativity

Einstein did not use the field equations of General Relativity to predict the perihelion precession of Mercury. He used the earliest Post Newtonian expansion of General Relativity to achieve this. The first Parameterised Post Newtonian (PPN) approximation of General Relativity was produced by Sir Arthur Eddington in 1922. It was extended by Ken Nordtvedt in 1969 and further by Clifford Will in 1972. It gives an approximation to General Relativity giving almost identical results. However whereas GR predicts movements in terms of the geodesics of curved space time (it doesn't provide forces and accelerations in the way that Newton's equation does) the PPN approximation predicts accelerations like Newton, and so it is effectively a correction to Newton's maths.

The acceleration due to gravity in a two body situation is given in the PPN equation below:

$$\text{Acceleration} = -\frac{GM}{r^2}\vec{n} - \frac{GM}{C^2 r^2}\left\{\left(V^2 - \frac{4GM}{r}\right)\vec{n} - 4\dot{r}\vec{v}\right\}$$

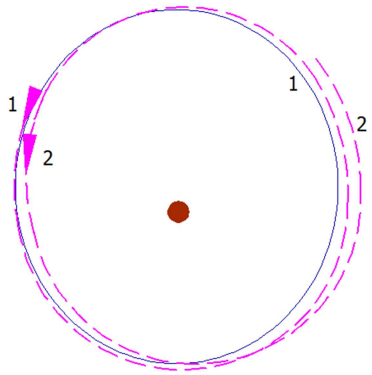
For circular orbits the above equation approximates to the one below and so it can easily be seen that the force of gravity reduces with velocity: To give an idea of the scale of the reduction, Mercury travels at approximately 58.98km/s at the perihelion and so this equates to a reduction in gravitational acceleration of 1.000000116.

$$\text{Acceleration} = -\frac{GM}{r^2}\left(1 - \frac{3V^2}{C^2}\right)$$

In section 7.3 of his book, "Theory and Experiment In Gravitational Physics" Clifford Will derives the equation for the precession of elliptical orbits using the PPN approximation.

So the approximation predicts that the gravitational potential reduces with velocity, and leads to the precession equation which provides the correct magnitude for elliptical precessions. But does its derivation ensure that the orbit is precessing rather than regressing ?

In the image below we have shown what we would expect to happen to an elliptical orbit in which a velocity related potential provides a decrease in force with velocity. This is the exact opposite of what we called the "Intuitive Solution" earlier.



Whilst the mathematics provides the correct magnitude, it is difficult to see why Einstein's velocity dependant potential arrives at a precession rather than a regression.

Comparing Theories

Simon Newcombe's correction approximately worked for Mercury but failed for the other inner planets. When we looked at the change in gravitational acceleration for Mercury, there was a decrease in force at the perihelion but a greater decrease in force at the aphelion and so there was a relative increase in acceleration at the perihelion in the order of 1.000000067 times greater than at the aphelion when compared to Newton.

If we do the same for the PPN equation but we use the PPN potential in the opposite way that it was intended (so the force increases with velocity rather than decreases) we find the ratio of the force increase is 1.000000065 times greater at the perihelion than at the aphelion. In other words, the ratio of force change to the Newton's force for both models would be almost identical.

When we compare the other planets (where Newcomb's modification didn't predict the correct precession) we find that the error in the predicted precession whilst far from identical, is similar to the error in the ratio of forces as per the table below:

Planet	Observed Precession Deviation From Newton Gravity	Precession Predicted by PPN/Einstein (Seconds of arc per century)	Precession Predicted by Newcomb $R^{2.00000016}$	Error Excess Of Newcomb Over PPN/Einstein	PPN Ratio Of Acceleration increase at Perihelion / Aphelion	Newcomb Ratio Of Acceleration increase at Perihelion / Aphelion	Error Excess Of Newcomb's ratio compared to Einstein Ratio
Mercury	**43.1** +/- 0.45	42.98	43.37	**+ 0.9%**	1.000000065	1.000000067	**+3%**
Venus	**8.4** +/- 4.8	8.61	16.98	**+ 97.2%**	1.000000001	1.000000002	**+100%**
Earth	**5.0** +/- 1.2	3.84	10.45	**+ 172%**	1.000000002	1.000000005	**+150%**
Mars	**1.36**	1.35	5.55	**+ 308%**	1.000000007	1.00000003	**+320%**

The change in the gravitational acceleration at Mercury's perihelion is approximately 1.00000005 for the PPN approximation and approximately 1.000002823 for Newcomb. ie Newcomb's change is 55 times larger than for PPN. The precession adheres much more closely to the ratio of force increase at the perihelion compared to the Aphelion than it does to the variation in the gravitational potential. It is quite likely that a model with a totally different velocity

dependant potential to GR could predict the correct precession of elliptical orbits as long as the ratio of increase in accelerations accords with those given by GR.

You might reasonably assume that the velocity related change in gravitational acceleration would have been tested to check that this extremely important part of General Relativity is correct. The problem is though that the variation in acceleration is very tiny. For example, the earth travels fastest at the perihelion when it is closest to the sun and this occurs round the 2nd of January each year. It travels most slowly at the aphelion at around the 2nd of July. The maximum velocity relative to the sun is 30.29km/s and the minimum velocity is 20.29km/s. This equates to a force/acceleration change ratio of only 1.000000016, ie 16 parts per billion.

You might also expect that this slight gravitational variation would show up as an apparent variation in G on Earth but again the answer is no. The various measurements of G over a number of decades have varied wildly with values ranging from about 6.670 x 10^{-11} m^3 kg^{-1} s^{-2} to 6.676 x 10^{-11} m^3 kg^{-1} s^{-2}. ie a massive variation approaching 1 part per thousand. (See web link http://www.nist.gov/pml/div684/102714-bigg.cfm) This lack of accuracy has caused growing concern as per the Big G conference at NIST (The National Institute for Standards & Technology in the US) on the attached link. There are a number of papers about this subject, one titled "Precision Measurement of the Newtonian Gravitational Constant Using Cold Atoms by G. Rosi et al, describes their measurement of G being to an accuracy of 150 PPM. I have been advised that an accuracy of 1PPM should be achievable within a decade using this type of experiment, but if Einstein's potential is accurate, a far greater level of precision will be required.

Another possibility might be to review the "flyby anomaly". This is the anomaly that has occurred when exploratory spacecraft have made flybys during slingshot manoeuvres. This could be an alternative way of investigating variations in gravitational acceleration associated with velocity with a greater level of precision. At present the anomalous Increase in gravitational effect has tended towards the higher Earth velocity at the perihelion but there are so few results and they vary quite considerably and so this gives no clear guide as to whether current models are correct, see the table below:

Approximate maximum Earth Velocity at Perigee is 30.29km/s and occurs on 2nd January
Approximate minimum Earth Velocity at Apogee is 29.29km/s and occurs on 2nd July
Approximate Mean Earth Velocity is 29.79km/s: and occurs on 2nd April and 2nd October

Spacecraft Project	**Galileo 1**	**NEAR**	**Cassini**	**Rosetta 1**	**Messenger**	**Rosetta 2**	**Rosetta 3**	**Juno**	**Hayabusa**
Date Of Flyby	**8th December 1990**	**23rd January 1998**	**18th August 1999**	**4th March 2005**	**2nd August 2005**	**13th November 2007**	**13th November 2009**	**9th October 2013**	**3rd December 2015**
Speed Increase at Perigee mm/s	2.56 +/- 0.05	7.21 +/- 0.07	-1.7 +/-0.9	0.67 +/- 0.02	0.008 +/-0.004	≈ 0.00	-0.004 +/-0.044	≈ 0.00	≈ 0.00 (less than 1.0mm/s)

Conclusion

With the exception of General Relativity, the possible causes of Mercury's precession can be understood intuitively as can the models of Leverrier, Paul Gerber, Asaph Hall and Simon Newcomb. For any model to predict the precession of an elliptical orbit, you would expect the force of gravity compared to Newton's force, to increase proportionately at the perihelion compared to the aphelion. This can be caused by the existence of inner planets or outer planets subject to suitable relative orbit positions or by applying a velocity dependant gravitational potential in which the strength of gravity increases with velocity. Gerber's model was the exception to this rule in that it predicted that the orbit would swing out slightly towards the perihelion and then in again when leaving the perihelion. By doing this, it changed the angle at the start of each new orbit.

The gravitational potential in General Relativity reduces with velocity. Given this we would have expected it to predict a regression of any elliptical orbit rather than a precession. We are unable to explain therefore why GR was able to arrive at the correct precession equation.

General Relativity has been tested in many ways to a high level of precision, although some of these tests have been a little obscure. However this velocity dependant change in potential is probably the most fundamental change from Newton Gravity and yet it has never been directly tested. This must now be the most significant test needed in gravitational physics, and hopefully technology will enable such a test in the near future if it doesn't already.